Normal Forms and Pushdown Automata

Mridul Aanjaneya

Stanford University

July 19, 2012

Midterm Review

- No class on Thursday (07/26).
- Practice problems.
- Office hours on Thursday (07/26) (1PM - 4PM) and (5PM - 8PM) (Gates 104).
- Office hours on Monday (07/30) (6PM - 9PM) (Gates 104).

Epsilon Productions

- We can almost avoid using productions of the form $A \to \varepsilon$ (called ε-productions).
  - The problem is that ε cannot be in the language of any grammar that has no ε-productions.

Theorem

If L is a CFL, then $L - \{\varepsilon\}$ has a CFG with no ε-productions.

Nullable Symbols

- To eliminate ε-productions, we first need to discover the nullable symbols = variables A such that $A \Rightarrow^* \varepsilon$.
- **Basis:** If there is a production $A \to \varepsilon$, then A is nullable.
- **Induction:** If there is a production $A \to \alpha$, and all symbols of α are nullable, then A is nullable.

Example: Nullable Symbols

$$S \rightarrow AB,\ A \rightarrow aA \mid \varepsilon,\ B \rightarrow bB \mid A$$

- **Basis:** A is nullable because of $A \rightarrow \varepsilon$.
- **Induction:** B is nullable because of $B \rightarrow A$.
- Then, S is nullable because of $S \rightarrow AB$.

Proof of Algorithm: Nullable Symbols

- Proof is very much like that for the algorithm for testing variables that derive terminal strings.
- Left to the imagination!

Eliminating ε-productions

- **Key idea:** turn each production

$$A \rightarrow X_1 \ldots X_n$$

into a family of productions.

- For each subset of nullable X's, there is one production with those eliminated from the right side in advance.
  - Except, if all X's are nullable, do not make a production with ε as the right hand side.

Example: Eliminating ε-productions

$$S \rightarrow ABC,\ A \rightarrow aA \mid \varepsilon,\ B \rightarrow bB \mid \varepsilon,\ C \rightarrow \varepsilon$$

- A, B, C and S are all nullable.
- New grammar:

$$S \rightarrow \cancel{ABC} \mid AB \mid \cancel{AC} \mid \cancel{BC} \mid A \mid B \mid \cancel{C}$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid b$$

- **Note:** C is now useless, eliminate its productions.

Why It Works

- Prove that for all variables A:
    1. If $w \neq \varepsilon$ and $A \Rightarrow^*_{\text{old}} w$, then $A \Rightarrow^*_{\text{new}} w$.
    2. If $A \Rightarrow^*_{\text{new}} w$, then $w \neq \varepsilon$ and $A \Rightarrow^*_{\text{old}} w$.
- Then, letting A be the start symbol proves that $L(\text{new}) = L(\text{old}) - \{\varepsilon\}$.
- (1) is an induction on the number of steps by which A derives w in the old grammar.

Proof of 1 - Basis

- If the old derivation is one step, then $A \rightarrow w$ must be a production.
- Since $w \neq \varepsilon$, this production also appears in the new grammar.
- Thus, $A \Rightarrow_{new} w$.

Proof of 1 - Induction

- Let $A \Rightarrow^*_{\text{old}} w$ be an n-step derivation, and assume the **IH** for derivations of less than n steps.
- Let the first step be $A \Rightarrow_{\text{old}} X_1 \ldots X_n$.
- Then w can be broken into $w = w_1 \ldots w_n$, where $X_i \Rightarrow^*_{\text{old}} w_i$, for all i, in fewer than n steps.

Proof of 1 - Induction

- By the **IH**, if $w_i \neq \varepsilon$, then $X_i \Rightarrow^*_{\text{new}} w_i$.
- Also, the new grammar has a production with A on the left, and just those X_i's on the right such that $w_i \neq \varepsilon$.
    - **Note:** They all cannot be ε, because $w \neq \varepsilon$.
- Follow a use of this production by the derivations $X_i \Rightarrow^*_{\text{new}} w_i$ to show that A derives w in the new grammar.

Proof of Converse

- We also need to show part (2) - if w is derived from A in the new grammar, then it is also derived in the old.
- Induction on number of steps in the derivation.
- Left as exercise.

Unit Productions

- A unit production is one whose right hand side consists of exactly one variable.
- These productions can be eliminated.
- **Key idea:** If $A \Rightarrow^* B$ by a series of unit productions, and $B \rightarrow \alpha$ is a non-unit production, then add the production $A \rightarrow \alpha$.
- Then drop all unit productions.

Unit Productions

- Find all pairs (A,B) such that $A \Rightarrow^* B$ by a sequence of unit productions only.
- **Basis:** Surely (A,A).
- **Induction:** If we have found (A,B), and $B \rightarrow C$ is a unit production, then add (A,C).

Proof that we find exactly the right pairs

- By induction on the order in which pairs (A,B) are found, we can show $A \Rightarrow^* B$ by unit productions.
- Conversely, by induction on the number of steps in the derivation by unit productions of $A \Rightarrow^* B$, we can show that the pair (A,B) is discovered.
- Left as exercises.

Proof: Unit Production Elimination Algorithm

- **Basic idea:** there is a leftmost derivation $A \Rightarrow^*_{lm} w$ in the new grammar if and only if there is such a derivation in the old.
- A sequence of unit productions and a non-unit production is collapsed into a single production of the new grammar.

Recap: Useless Symbols

- A symbol is useful if it appears in some derivation of some terminal string from the start symbol.
- Otherwise it is useless. Eliminate all useless symbols by:
  1. Eliminating symbols that derive no terminal string.
  2. Eliminating unreachable symbols.

Cleaning Up a Grammar

Theorem

If L is a CFL, then there is a CFG for L - $\{\varepsilon\}$ that has:

- No useless symbols.
- No ε-productions.
- No unit productions.

- i.e., every right side is either a single terminal or has length ≥ 2.

Cleaning Up

- **Proof:** Start with a CFG for L.
- Perform the following steps in order:
  1. Eliminate ε-productions.
  2. Eliminate unit productions.
  3. Eliminate variables that derive no terminal string.
  4. Eliminate variables not reachable from the start symbol.
- **Note:** (1) can create unit productions or useless variables, so it must come first.

Chomsky Normal Form

Definition

A CFG is said to be in Chomsky Normal Form if every production is of one of these two forms:

- $A \rightarrow BC$ (right side is two variables).
- $A \rightarrow a$ (right side is a single terminal).

Theorem

If L is a CFL, then L - $\{\varepsilon\}$ has a CFG in CNF.

Proof of CNF Theorem

- **Step 1:** Clean the grammar, so every production right side is either a single terminal or of length at least 2.
- **Step 2:** For each right side $\neq$ a single terminal, make the right side all variables.
  1. For each terminal a create a new variable A_a and production $A_a \rightarrow a$.
  2. Replace a by A_a in right sides of length > 2.

Example: Step 2

- Consider production $A \rightarrow BcDe$.
- We need variables A_c and A_e with productions $A_c \rightarrow c$ and $A_e \rightarrow e$.
  - **Note:** you create at most one variable for each terminal, and use it everywhere it is needed.
- Replace $A \rightarrow BcDe$ by $A \rightarrow BA_cDA_e$.

Proof of CNF Theorem

- **Step 3:** Break right sides longer than 2 into a chain of productions with right sides of two variables.
- $A \rightarrow BCDE$ is replaced by $A \rightarrow BF$, $F \rightarrow CG$ and $G \rightarrow DE$.
  - **Note:** F and G must be used nowhere else.
- In the new grammar, $A \Rightarrow BF \Rightarrow BCG \Rightarrow BCDE$.
- **More importantly:** Once we choose to replace A by BF, we must continue to BCG and BCDE.
  - Because F and G have only one production.

Proof of CNF Theorem (Formally)

- We must prove that Steps 2 and 3 produce new grammars whose languages are the same as the previous grammar.
- Proofs are of a familiar type and involve inductions on the lengths of derivations.
    - Left as exercises.

Pushdown Automata

- A PDA is an automaton equivalent to the CFG in language-defining power.
- Only the nonterministic PDA's define all possible CFL's.
- But the deterministic version models parsers.
  - Most programming languages have deterministic PDA's.

Intuition: PDA

- Think of an ε-NFA with the additional power that it can manipulate a stack.
- Its moves are determined by:
  1. The current state (of its NFA).
  2. The current input symbol (or ε), and
  3. The current symbol on top of its stack.

Intuition: PDA

- Being nondeterministic, the PDA can have a choice of next moves.
- In each choice, the PDA can:
  1. Change state, and also
  2. Replace the top symbol on the stack by a sequence of zero or more symbols.
     - Zero symbols = pop.
     - Many symbols = sequence of pushes.

PDA Formalism

- A PDA is described by:
  1. A finite set of states (Q, typically).
  2. An input alphabet (Σ, typically).
  3. A stack alphabet (Γ, typically).
  4. A transition function (δ, typically).
  5. A start state (q_0, in Q, typically).
  6. A start symbol (Z_0, in Γ, typically).
  7. A set of final states ($F \subseteq Q$, typically).

Conventions

- a, b,. . . are input symbols.
  - But sometimes we allow ε as a possible value.
- . . ., X, Y, Z are stack symbols.
- . . ., w, x, y, z are strings of input symbols.
- α, β,. . . are strings of stack symbols.

The Transition Function

- Takes three arguments:
  1. A state in Q.
  2. An input which is either a symbol in Σ or ε.
  3. A stack symbol in Γ.
- δ(q,a,Z) is a set of zero or more actions of the form (p,α).
  - p is a state, α is a string of stack symbols.

Actions of the PDA

- If δ(q,a,Z) contains (p,α) among its actions, then one thing the PDA can do in state q, with a at the front of the input, and Z on top of the stack is:
  1. Change the state to p.
  2. Remove a from the front of the input (but a may be ε).
  3. Replace Z on the top of the stack by α.

Example: PDA

- Design a PDA to accept $\{0^n1^n \mid n \geq 1\}$.
- The states:
  - q = start state. We are in state q if we have seen only 0's so far.
  - p = we've seen at least one 1 and may now proceed only if the inputs are 1's.
  - f = final state; accept.

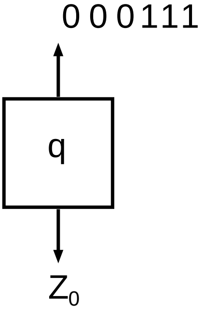
Example: PDA

- The stack symbols:
  - Z_0 = start symbol. Also marks the bottom of the stack, so we know we have counted the same number of 1's as 0's.
  - X = marker, used to count the number of 0's seen on the input.

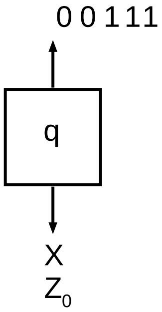
Example: PDA

- The transitions:
  - $\delta(q,0,Z_0) = \{(q,XZ_0)\}$.
  - $\delta(q,0,X) = \{(q,XX)\}$. These two rules cause one X to be pushed onto the stack for each 0 read from the input.
  - $\delta(q,1,X) = \{(p,\varepsilon)\}$. When we see a 1, go to state p and pop one X.
  - $\delta(p,1,X) = \{(p,\varepsilon)\}$. Pop one X per 1.
  - $\delta(p,\varepsilon,Z_0) = \{(f,Z_0)\}$. Accept at bottom.

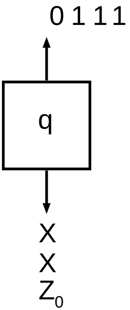
Actions of the Example PDA



Actions of the Example PDA



Actions of the Example PDA



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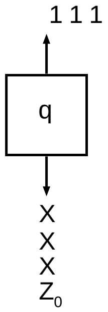

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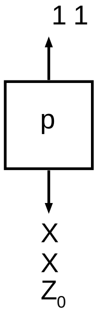

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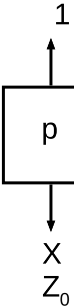

Actions of the Example PDA