Homomorphisms and Efficient State Minimization

Mridul Aanjaneya

Stanford University

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Homomorphisms

- A homomorphism on an alphabet is a function that gives a string for each symbol in that alphabet.
- **Example:** $h(0) = ab$, $h(1) = \varepsilon$.
- Extend to strings by $h(a_1a_2 \ldots a_n)=h(a_1)h(a_2)\ldots h(a_n)$.
- **Example:** $h(01010) = ababab$.

Closure Properties: Homomorphism

- If L is regular, and h is a homomorphism on its alphabet, then $h(L) = \{h(w) \mid w \in L\}$ is also regular.
- **Proof:** Let E be a regular expression for L.
- Apply h to each symbol in E.
- Language of resulting RE is $h(L)$.

Closure under Homomorphism: Example

- Let $h(0) = ab$, $h(1) = \varepsilon$.
- Let L be the language of a regular expression $01^* + 10^*$.
- Then $h(L)$ is the language of regular expression $ab\varepsilon^* + \varepsilon(ab)^*$.
- **Note:** $ab\varepsilon^* + \varepsilon(ab)^* = ab + (ab)^* = (ab)^*$.

Inverse Homomorphisms

- Let h be a homomorphism and L a language whose alphabet is the output language of h.
- $h^{-1}(L) = \{w \mid h(w) \in L\}$.

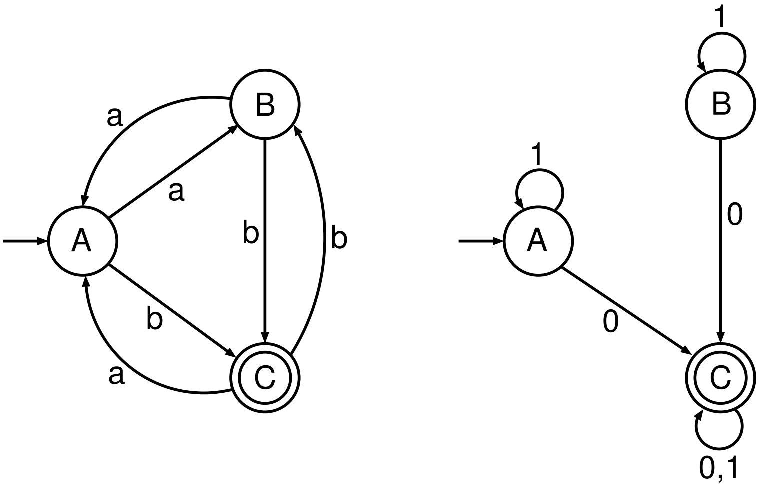
Inverse Homomorphisms: Example

- Let $h(0) = ab$, $h(1) = \varepsilon$.
- Let $L = \{abab, baba\}$.
- $h^{-1}(L) =$ the language with two 0's and any number of 1's $=$ $L(1^*01^*01^*)$.
- **Note:** No string maps to baba, any string with exactly two 0's maps to abab.

Closure Proof for Inverse Homomorphism

- Start with a DFA A for L.
- Construct a DFA B for $h^{-1}(L)$ with:
  - The same set of states.
  - The same start state.
  - The same final states.
  - Input alphabet = the symbols to which the homomorphism h applies.
- The transitions for B are computed by applying h to an input symbol a and seeing where A would go on sequence of input symbols h(a).
- Formally, $\delta_B(q,a) = \delta_A(q,h(a))$.

Inverse Homomorphism Construction: Example



- $h(0) = ab$, $h(1) = \varepsilon$.

Closure Proof for Inverse Homomorphism

- Induction on $|w|$ shows that $\delta_B(q_0, w) = \delta_A(q_0, h(w))$.
- **Basis:** $w = \varepsilon$.
  $\delta_B(q_0, \varepsilon) = q_0$, and
  $\delta_A(q_0, h(\varepsilon)) = \delta_A(q_0, \varepsilon) = q_0$.
- **Inductive Step:** Let $w = xa$, assume **IH** for x.
- $\delta_B(q_0, w) = \delta_B(\delta_B(q_0, x), a) = \delta_B(\delta_A(q_0, h(x)), a)$ (from **IH**)
- $= \delta_A(\delta_A(q_0, h(x)), h(a))$ (by definition of B)
- $= \delta_A(q_0, h(x)h(a))$ (by definition of extended δ_A)
- $= \delta_A(q_0, h(w))$ (by definition of h)

Decision Property: Equivalence

- Given regular languages L and M, is L = M?
- Algorithm involves constructing the product DFA P from DFA's for L and M.
- Make the final states of P be those states [q,r] such that exactly one of q and r is a final state of its own DFA.
- Thus, P accepts w iff w is in exactly one of L and M.
- The product DFA's language is empty iff L = M.

- **Note:** We already have a better algorithm to test emptiness.

Minimum State DFA

- In principle, since we can test for equivalence of DFA's we can, given a DFA A find the DFA with the fewest states accepting $L(A)$.
- Test all smaller DFA's for equivalence with A.
- But thats a terrible algorithm.

Efficient State Minimization

- Construct a table with all pairs of states.
- If you find a string that distinguishes two states (takes exactly one to an accepting state), mark that pair.
- Algorithm is a recursion on the length of the shortest distinguishing string.

Efficient State Minimization

- **Basis:** Mark a pair if exactly one is a final state.
- **Induction:** Mark [q,r] if there is some input symbol a such that [δ(q,a),δ(r,a)] is marked.
- After no more marks are possible, the unmarked states are equivalent and can be merged into one state.

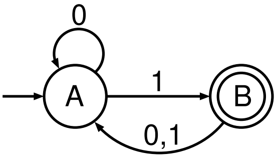
Constructing the Minimum State DFA

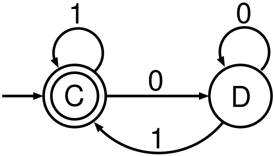
- Suppose $q_1, \ldots, q_k$ are indistinguishable states.
- Replace them by one state q.
- Then $\delta(q_1, a), \ldots, \delta(q_k, a)$ are all indistinguishable states, otherwise, we should have marked at least one more pair.
- Let $\delta(q, a) =$ the representative state for that group.

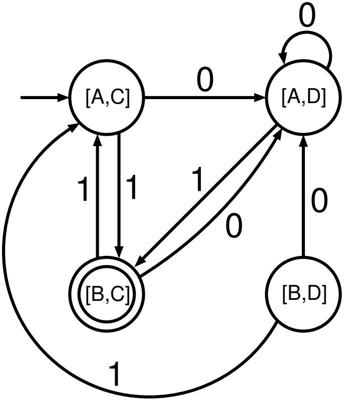
Eliminating Indistinguishable States

- Unfortunately, combining indistinguishable states could leave us with unreachable states in the minimum-state DFA.
- Thus, before or after, remove states that are not reachable from the start state.

Example: State Minimization







Transitivity of "Indistinguishable"

Proposition

If state p is indistinguishable from q, and q is indistinguishable from r, then p is indistinguishable from r.

- **Proof:** The outcome (accept or don't) of p and q on input w is the same, and the outcome of q and r on w is the same, then likewise the outcome of p and r.

Clincher

- We have combined states of the given DFA wherever possible.
- Could there be another, completely unrelated DFA with fewer states?
- No. The proof involves minimizing the DFA we derived with the hypothetical better DFA.

Proof: No Unrelated, Smaller DFA

- Let A be our minimized DFA; let B be a smaller equivalent.
- Consider an automaton with the states of A and B combined.
- Use distinguishable in its contrapositive form:
  - If states q and p are indistinguishable, so are δ(q,a) and δ(p,a).

Inductive Hypothesis

Hypothesis

Every state q of A is indistinguishable from some state of B.

- Induction is on the length of the shortest string taking you from the start state of A to q.

Induction

- **Basis:** Start states of A and B are indistinguishable, because $L(A) = L(B)$.
- **Induction:** Suppose $w = xa$ is a shortest string getting A to state q.
- By the **IH**, x gets A to some state r that is indistinguishable from some state p of B.
- Then $\delta(r,a) = q$ is indistinguishable from $\delta(p,a)$.

Induction

- **Key idea:** Two states of A cannot be indistinguishable from the same state of B, or they would be indistinguishable from each other.
    - But A is already a minimum state DFA!
- Thus, B has at least as many states as A.

Fibonacci Numbers and the Golden Ratio

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
\Rightarrow F_n &= \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}
\end{aligned}
$$