Pumping Lemma and Closure Properties of CFL's

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Pumping Lemma for CFL's: Intuition

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could pump the cycle and discover an infinite sequence of strings that had to be in the language.

Pumping Lemma for CFL's: Intuition

- For CFL's the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to pump in tandem.
- That is, if we repeat each of these two pieces the same number of times, we get another string in the language.

The CFL Pumping Lemma

Theorem

For every CFL L there is an integer n, such that for every string z in L of length $\geq n$, there exists $z = uvwxy$ such that:

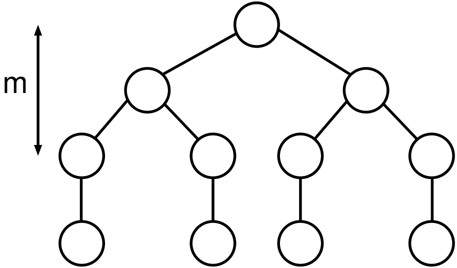
- $|vwx| \leq n$.
- $|vx| > 0$.
- For all $i \geq 0$, $uv^iwx^iy \in L$.

Proof of the Pumping Lemma

- Start with a CNF grammar for L - $\{\varepsilon\}$.
- Let the grammar have m variables.
- Pick n $= 2^m$.
- Let $|z| \geq n$.
- We claim ("Lemma 1") that a parse tree with yield z must have a path of length m+2 or more.

Proof of Lemma 1

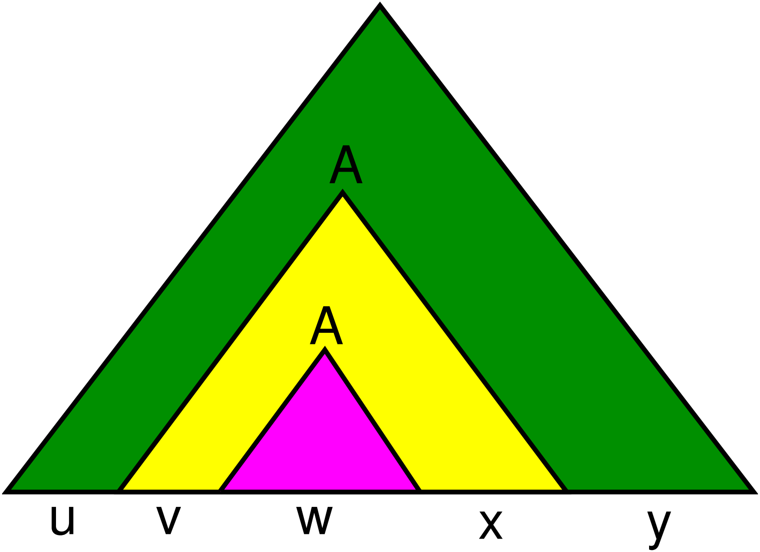
- If all the paths in the parse tree of a CNF grammar are of length $\leq m+1$, then the longest yield has length 2^{m-1}, as in:



Proof of the Pumping Lemma

- Now we know that the parse tree for z has a path with at least $m+1$ variables.
- Consider some longest path.
- There are only m different variables, so among the lowest $m+1$ we can find two nodes with the same label, say A.
- The parse tree thus looks like:

Proof of the Pumping Lemma



Using the Pumping Lemma

- Non-CFL's typically involve trying to match two pairs of counts or match two strings.
- **Example:** Show that $L = \{0^i10^i10^i \mid i \geq 1\}$ is not a CFL.
- Proof using the pumping lemma.
- Suppose L were a CFL.
- Let n be L's pumping length.

Using the Pumping Lemma

- Consider $z = 0^n 1 0^n 1 0^n$.
- We can write $z = uvwxy$, where $|vwx| \leq n$, and $|vx| \geq 1$.
- **Case 1:** vx has no 0's.
  - Then at least one of them is a 1, and uwy has at most one 1, which no string in L does.

Using the Pumping Lemma

- Still considering $z = 0^n10^n10^n$.
- **Case 2:** vx has at least one 0.
  - vwx is too short (length $\leq n$) to extend to all three blocks of 0's in $0^n10^n10^n$.
  - Thus, uwy has at least one block of n 0's, and at least one block with fewer than n 0's.
  - Thus, uwy is not in L.

Summary of Decision Properties

- As usual, when we talk about a CFL we really mean a representation for the CFL, e.g., a CFG or a PDA accepting by final state or empty stack.
- There are algorithms to decide if:
  - String w is in CFL L.
  - CFL L is empty.
  - CFL L is infinite.

Non-decision Properties

- Many questions that can be decided for regular sets cannot be decided for CFL's.
- **Example:** Are two CFL's the same?
- **Example:** Are two CFL's disjoint?
  - How would you do that for regular languages?
- Need theory of Turing Machines and decidability to prove no algorithm exists.

Testing Emptiness

- We already did this.
- We learned to eliminate variables that generate no terminal string.
- If the start symbol is one of these, then the CFL is empty; otherwise not.

Testing Membership

- Want to know if string w is in L(G).
- Assume G is in CNF.
    - Or convert the given grammar to CNF.
    - w $= \varepsilon$ is a special case, solved by testing if the start symbol is nullable.
- Algorithm CYK is a good example of dynamic programming and runs in $O(n^3)$, where n $= |w|$.

CYK Algorithm

- Let $w = a_1a_2 \ldots a_n$.
- We construct an n-by-n triangular array of sets of variables.
- $X_{ij} = \{\text{variables } A \mid A \Rightarrow^* a_i \ldots a_j\}$.
- Induction on $j\text{-}i+1$.
  - The length of the derived string.
- Finally, ask if S is in X_{1n}.

CYK Algorithm

- **Basis:** $X_{ii} = \{A \mid A \rightarrow a_i$ is a production$\}$.
- **Induction:** $X_{ij} = \{A \mid$ there is a production $A \rightarrow BC$ and an integer k, with $i \leq k < j$, such that B is in X_{ik} and C is in $X_{k+1,j}\}$.

Example: CYK Algorithm

- Grammar: $S \rightarrow AB$, $A \rightarrow BC|a$, $B \rightarrow AC|b$, $C \rightarrow a|b$
- String w = ababa.
- X_{11} = {A,C}, X_{22} = {B,C}, X_{33} = {A,C}, X_{44} = {B,C}, X_{55} = {A,C}.
- X_{12} = {B,S}, X_{23} = {A}, X_{34} = {B,S}, X_{45} = {A}.
- X_{13} = {A}, X_{24} = {B,S}, X_{35} = {A}.
- X_{14} = {B,S}, X_{25} = {A}.
- X_{15} = {A}.

Testing Infiniteness

- The idea is essentially the same as for regular languages.
- Use the pumping length n.
- If there is a string in the language of length between n and 2n-1, then the language is infinite; otherwise not.

Closure Properties of CFL's

- CFL's are closed under union, concatenation, and Kleene closure.
- Also, under reversal, homomorphisms and inverse homomorphisms.
- But **not** under intersection or difference.

Closure of CFL's under Union

- Let L and M be CFL's with grammars G and H, respectively.
- Assume G and H have no variables in common.
  - Names of variables do not affect the language.
- Let S_1 and S_2 be the start symbols of G and H.

Closure of CFL's under Union

- Form a new grammar for $L \cup M$ by combining all the symbols and productions of G and H.
- Then, add a new start symbol S.
- Add the production $S \rightarrow S_1 | S_2$.

Closure of CFL's under Union

- In the new grammar, all derivations start with S.
- The first step replaces S by either S_1 or S_2.
- In the first case, the result must be a string in $L(G) = L$, and in the second case a string in $L(H) = M$.

Closure of CFL's under Concatenation

- Let L and M be CFL's with grammars G and H, respectively.
- Assume G and H have no variables in common.
- Let S_1 and S_2 be the start symbols of G and H.

Closure of CFL's under Concatenation

- Form a new grammar for LM by combining all the symbols and productions of G and H.
- Add a new start symbol S.
- Add the production $S \rightarrow S_1S_2$.
- Every derivation from S results in a string in L followed by one in M.

Closure under Star

- Let L have grammar G, with start symbol S_1.
- Form a new grammar for L^* by introducing to G a new start symbol S and the productions $S \to S_1S \mid \varepsilon$.
- A rightmost derivation from S generates a sequence of zero or more S_1's, each of which generates some string in L.

Closure of CFL's under Reversal

- If L is a CFL with a grammar G, form a grammar for L^R by reversing the right side of every production.
- **Example:** Let G have $S \rightarrow 0S1 \mid 01$.
- The reversal of $L(G)$ has grammar $S \rightarrow 1S0 \mid 10$.

Closure of CFL's under Homomorphisms

- Let L be a CFL with a grammar G.
- Let h be a homomorphism on the terminal symbols of G.
- Construct a grammar for h(L) by replacing each terminal symbol a by h(a).

Example: Closure under Homomorphisms

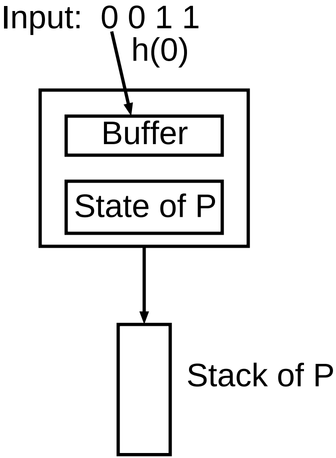
- G has productions $S \rightarrow 0S1 \mid 01$.
- h is defined by $h(0) = ab$, $h(1) = \varepsilon$.
- $h(L(G))$ has the grammar with productions $S \rightarrow abS \mid ab$.

Closure under Inverse Homomorphisms

- Here, grammars do not help us.
- But a PDA construction serves nicely.
- **Intuition:** Let $L = L(P)$ for some PDA P.
- Construct PDA P' to accept $h^{-1}(L)$.
- P' simulates P, but keeps, as one component of a two-component state a buffer that holds the result of applying h to one input symbol.

Architecture of P'

- Read first remaining symbol in buffer as if it were input to P.



Formal Construction of P'

- States are pairs [q,b], where:
  1. q is a state of P.
  2. b is a suffix of h(a) for some symbol a.
- Thus, only a finite number of possible values for b.
- Stack symbols of P' are those of P.
- Start state of P' is [q_0,ε].

Formal Construction of P'

- Input symbols of P' are the symbols to which h applies.
- Final states of P' are the states [q,ε] such that q is a final state of P.

Transitions of P'

1. $\delta'([q,\varepsilon],a,X) = \{([q,h(a)],X)\}$ for any input symbol a of P' and any stack symbol X.
   - When the buffer is empty, P' can reload it.
2. $\delta'([q,bw],\varepsilon,X)$ contains $([p,w],\alpha)$ if $\delta(q,b,X)$ contains (p,α), where b is either an input symbol of P or ε.
   - Simulate P from the buffer.

Proving Correctness of P'

- We need to show that L(P') $=$ h^{-1}(L(P)).
- **Key argument:** P' makes the transition ([q,ε],w,Z_0) $\vdash^*$ ([q,x],ε,α) if and only if P makes transition (q_0,y,Z_0) $\vdash^*$ (q,ε,α), h(w) $=$ yx, and x is a suffix of the last symbol of w.
- Proof in both directions is an induction on the number of moves made.
  - Left as exercises.

Nonclosure under Intersection

- Unlike the regular languages, the class of CFL's is not closed under intersection.
- We know that $L_1 = \{0^n 1^n 2^n \mid n \geq 1\}$ is not a CFL (using the pumping lemma).
- However, $L_2 = \{0^n 1^n 2^i \mid n \geq 1, i \geq 1\}$ is.
  - CFG: $S \rightarrow AB$, $A \rightarrow 0A1 | 01$, $B \rightarrow 2B | 2$.
- So is $L_3 = \{0^i 1^n 2^n \mid n \geq 1, i \geq 1\}$.
- But $L_1 = L_2 \cap L_3$.

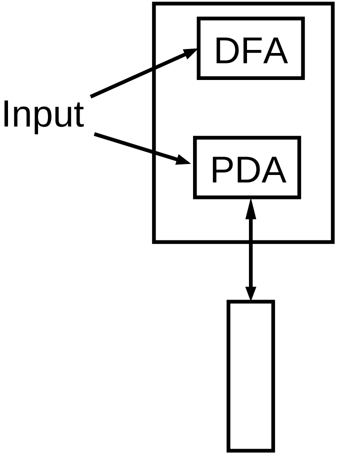
Nonclosure under Difference

- We can prove something more general:
  - Any class of languages that is closed under difference is closed under intersection.
- **Proof:** $L \cap M = L - (L - M)$.
- Thus, if CFL's were closed under difference, they would be closed under intersection, but they are not.

Intersection with a Regular Language

- Intersection of two CFL's need not be context-free.
- But the intersection of a CFL with a regular language is always a CFL.
- Proof involves running a DFA in parallel with a PDA, and noting that the combination is a PDA.
  - PDA's accept by final state.

PDA and DFA in parallel



Formal Construction

- Let the DFA A have transition function δ_A.
- Let the PDA P have transition function δ_P.
- States of combined PDA are [q,p], where q is a state of A and p is a state of P.
- δ([q,p],a,X) contains ([δ_A(q,a),r],α) if δ_P(p,a,X) contains (r,α).
  - **Note:** a could be ε, in which case $\delta_A(q,a) = q$.

Formal Construction

- Accepting states of combined PDA are those [q,p] such that q is an accepting state of A and p is an accepting state of P.
- **Easy induction:** $([q_0,p_0],w,Z_0) \vdash^* ([q,p],\varepsilon,\alpha)$ if and only if $\delta_A(q_0,w) = q$ and in P: $(p_0,w,Z_0) \vdash^* (p,\varepsilon,\alpha)$.