CS 323: Numerical Analysis and Computing

MIDTERM #1

Instructions: This is an **open notes** exam, i.e., you are allowed to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are **not permitted** to use laptop computers, cell phones, tablets, or any other hand-held electronic devices.

Name	
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Part #1	
Part #2	
Part #3	
Part #4	
Part #5	
TOTAL	

1. [24% = 4 questions $\times$ 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

   (a) Which of the following statements regarding the cost of methods for solving an $n \times n$ linear system $Ax = b$ are true?

      i. The cost of computing the LU factorization is generally proportional to n^2.

      ii. The cost of backward substitution on a dense upper triangular matrix is generally proportional to n^2.

      iii. If a matrix A has no more than 3 non-zero entries per row, the cost of each iteration of the Jacobi method is proportional to n.

   (b) If an $n \times n$ matrix A is poorly conditioned (i.e., it has a very large condition number), then

      i. Solving $Ax = b$ would be difficult with LU decomposition or Gaussian elimination, but iterative methods (Jacobi, Gauss-Seidel) would not have a problem.

      ii. Solving $Ax = b$ accurately with iterative methods (Jacobi, Gauss-Seidel) would be difficult, but LU decomposition with pivoting would not have a problem.

      iii. Solving $Ax = b$ accurately will be challenging regardless of the method we use.

   (c) Consider the rectangular $m \times n$ matrix A (with $m > n$), and the vector $b \in \mathbb{R}^m$. If x is the *least squares solution* to $Ax \approx b$, can we say that x is an actual solution to $Ax = b$?

      i. Yes, in fact $Ax = b$ has many solutions and the *least squares solution* is the one with the smallest L_2-norm of the residual vector $\|r\|_2$.

      ii. No, the system $Ax = b$ will generally not have a solution. What we call the *least squares solution* is the vector x with the smallest L_2-norm of the error vector $\|x - x_{\text{exact}}\|_2$.

      iii. No, the system $Ax = b$ will generally not have a solution. What we call the *least squares solution* is the vector x with the smallest L_2-norm of the residual vector $\|b - Ax\|_2$.

   (d) Which of the following methods can be used for solving the system $Ax = b$, where A is a symmetric, diagonally dominant, square $n \times n$ matrix?

      i. LU factorization with full pivoting.

      ii. System of normal equations.

      iii. Gauss-Seidel method.

      iv. Jacobi method.

2. [18% = 3 questions × 6% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

(a) Consider the following matrix A whose LU factorization we wish to compute using Gaussian elimination:

$$A = \begin{bmatrix} 4 & -8 & 1 \\ 6 & 5 & 7 \\ 0 & -10 & -3 \end{bmatrix}$$

What will be the initial pivot element if (no explanation required)

- No pivoting is used?
  Answer: 4
- Partial pivoting is used?
  Answer: 6
- Full pivoting is used?
  Answer: -10

(b) State one defining property of a *singular* matrix A. Suppose that the linear system $Ax = b$ has two distinct solutions x and y. Use the property you gave to prove that A must be singular.

Answer: A matrix A is singular when the system $Ax = 0$ has a solution other than $x = 0$. If $Ax = b$ and $Ay = b$, subtracting the two equations gives $A(x - y) = 0$. Since x and y are two distinct vectors, $x - y \neq 0$, implying that the matrix is singular.

(c) Mention one advantage of the Gauss-Seidel algorithm over the Jacobi algorithm and one disadvantage.

- *Advantage:* Gauss-Seidel algorithm has a faster convergence rate compared to the Jacobi algorithm.
- *Disadvantage:* Gauss-Seidel algorithm is difficult to parallelize compared to the Jacobi algorithm.

3. [14%] Consider the five points:

$$\begin{aligned}
(x_1, y_1) &= (-3, -1) \\
(x_2, y_2) &= (-2, 1) \\
(x_3, y_3) &= (0, 2) \\
(x_4, y_4) &= (1, 3) \\
(x_5, y_5) &= (3, 2)
\end{aligned}$$

(a) We want to determine a straight line $y = c_1 x + c_0$ that approximates these points as closely as possible, in the least squares sense. Write a least squares system $Ax \approx b$ which can be used to determine the coefficients c_1 and c_0.

Answer: The least squares system is given below:

$$\begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

(b) Solve this least squares system, using the method of normal equations.

Answer: The normal equations for the above system are given below:

$$\begin{bmatrix} 23 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

Solving this system of equations gives $c_1 = 0.5, c_0 = 1.5$.

4. [18%] The general form of an iterative method for solving the system $Ax = b$ has the form

$$x^{(k)} = Tx^{(k-1)} + c$$

where the matrix T and the vector c are such that the equation $x = Tx + c$ is equivalent to the original system $Ax = b$.

(a) If $x^\star$ is the *exact* solution of the system $Ax = b$, show that

$$x^{(k)} - x^\star = T\left(x^{(k-1)} - x^\star\right)$$

Answer: The following two equations hold for $x^\star, x^{(k)}$ and $x^{(k-1)}$:

$$\begin{aligned} x^{(k)} &= Tx^{(k-1)} + c \\ x^\star &= Tx^\star + c \end{aligned}$$

Subtracting the second equation from the first gives the required identity.

(b) If $r^{(k)} = b - Ax^{(k)}$ is the residual vector after the k^{th} iteration of the method, show that

$$r^{(k)} = ATA^{-1}r^{(k-1)}$$

Hint: Use the identity $r^{(k)} = -Ae^{(k)}$, or equivalently $e^{(k)} = -A^{-1}r^{(k)}$. Here, $e^{(k)} = x^{(k)} - x^\star$ is the *error vector* after the k^{th} iteration.
Answer: The identity in part (a) can be written in the equivalent form:

$$e^{(k)} = Te^{(k-1)}$$

Multiplying both sides of this equation by A gives,

$$r^{(k)} = Ae^{(k)} = ATe^{(k-1)}$$

Now using the fact that $e^{(k-1)} = A^{-1}r^{(k-1)}$ gives the required identity.

(c) Show that

$$r^{(k)} = AT^kA^{-1}r^{(0)}$$

Answer: This can be proved by repeated applications of the identity in part (b). For example:

$$\begin{aligned} r^{(k)} &= ATA^{-1}r^{k-1} = ATA^{-1}ATA^{-1}r^{k-2} \\ &= AT^2A^{-1}r^{k-2} = \ldots = AT^kA^{-1}r^{(0)} \end{aligned}$$

5. [26%] Consider the elimination matrix $M_k = I - m_k e_k^T$ and its inverse $L_k = I + m_k e_k^T$ used in the LU decomposition process, where

$$m_k = (0, \ldots, 0, m_{k+1}^{(k)}, \ldots, m_n^{(k)})^T$$

and e_k is the kth column of the identity matrix. Let $P^{(ij)}$ be the permutation matrix that results from swapping the i-th and j-th rows (or columns) of the identity matrix.

(a) [6%] Show that if $i, j > k$ then $L_k P^{(ij)} = P^{(ij)}(I + P^{(ij)} m_k e_k^T)$.

(b) [10%] Recall that the matrix L resulting from performing Gaussian elimination with partial pivoting is given by

$$L = P_1 L_1 \ldots P_{n-1} L_{n-1}$$

where the permutation matrix P_i permutes row i with some row i' where $i < i'$. Show that L can be rewritten as

$$L = P_1 \ldots P_{n-1} L_1^P \ldots L_{n-1}^P$$

where $L_k^P = I + (P_{n-1} \ldots P_{k+1} m_k) e_k^T$.

(c) [10%] Show that $L_1^P \ldots L_{n-1}^P$ is lower triangular.

Answer:

(a) The matrix $m_k e_k^T$ has non-zero elements only in the k^{th} column in the positions corresponding to rows $(k+1)$ through n. Additionally, $m_k e_k^T P^{(ij)}$ is the result of swapping the i^{th} and j^{th} columns of $m_k e_k^T$, which are both zero. Thus, $m_k e_k^T P^{(ij)} = m_k e_k^T$. Using this result, we have

$$\begin{aligned}
(I + m_k e_k^T) P^{(ij)} &= P^{(ij)} + m_k e_k^T P^{(ij)} \\
&= P^{(ij)} + m_k e_k^T \\
&= P^{(ij)} + P^{(ij)} P^{(ij)} m_k e_k^T \\
&= P^{(ij)} (I + P^{(ij)} m_k e_k^T)
\end{aligned}$$

The third equality follows because $P^{(ij)} P^{(ij)} = I$, the identity matrix.

(b) Let q_k be a vector containing non-zero entries only in the positions $(k+1)$ through n. Then, using part (a), we have

$$(I + q_k e_k^T) P_i = P_i (I + P_i q_k e_k^T) = P_i (I + \hat{q}_k e_k^T)$$

where the vector $\hat{q}_k = P_i q_k$ also has non-zero entries in the positions $(k+1)$ through n. Consequently, in the product $P_1 L_1 \ldots P_{n-1} L_{n-1}$, we can "propagate" each permutation matrix P_i (in increasing order of the index i) to the left of all matrices L_k with $k \leq i$, while changing each matrix L_k according to the equation above (multiplying its second term with P_i from the left). For example,

$$
\begin{aligned}
P_1 L_1 P_2 L_2 P_3 L_3 &= P_1 (I + m_1 e_1^T) P_2 (I + m_2 e_2^T) P_3 (I + m_3 e_3^T) \\
&= P_1 P_2 (I + P_2 m_1 e_1^T)(I + m_2 e_2^T) P_3 (I + m_3 e_3^T) \\
&= P_1 P_2 (I + P_2 m_1 e_1^T) P_3 (I + P_3 m_2 e_2^T)(I + m_3 e_3^T) \\
&= P_1 P_2 P_3 (I + P_3 P_2 m_1 e_1^T)(I + P_3 m_2 e_2^T)(I + m_3 e_3^T) \\
&= P_1 P_2 P_3 L_1^P L_2^P L_3^P
\end{aligned}
$$

where $L_k^P = I + P_{n-1} \dots P_{k+1} m_k e_k^T$. For the complete solution, this argument can be extended to arbitrary n.

(c) Each matrix L_k^P can be written as $L_k^P = I + \hat{q}_k e_k^T$, where $\hat{q}_k = P_{n-1} \dots P_{k+1} m_k$, like m_k, only has non-zero entries in the positions $(k+1)$ through n. Furthermore,

$$
\begin{aligned}
L_1^P L_2^P \dots L_{n-1}^P &= (I + \hat{q}_1 e_1^T)(I + \hat{q}_2 e_2^T) \dots (I + \hat{q}_{n-1} e_{n-1}^T) \\
&= I + \hat{q}_1 e_1^T + \hat{q}_2 e_2^T + \dots \hat{q}_{n-1} e_{n-1}^T
\end{aligned}
$$

since $e_i^T \hat{q}_j = 0$ for $i < j$, causing all the cross-terms $(\hat{q}_i e_i^T)(\hat{q}_j e_j^T)$ in the original product to vanish (for $i < j$). Since each term $\hat{q}_i e_i^T$ contributes non-zero entries only below the diagonal, the entire matrix $I + \hat{q}_1 e_1^T + \hat{q}_2 e_2^T + \dots \hat{q}_{n-1} e_{n-1}^T$ is lower triangular.