CS 323: Numerical Analysis and Computing

MIDTERM #1

Instructions: This is an open notes exam, i.e., you are allowed to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are not permitted to use laptop computers, cell phones, tablets, or any other hand-held electronic devices.

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1. [24% = 4 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

(a) Given a non-singular \( n \times n \) system of linear equations \( Ax = b \), the solution vector \( x \) remains unchanged after

(Circle the TWO most correct answers.)

i. Permuting the rows of \( A \) and \( b \).
ii. Permuting the columns of \( A \).
iii. Multiplying both sides of the equation from the left by a non-singular \( n \times n \) matrix \( M \).

(b) Let \( A \) be an \( m \times n \) matrix. Then the matrix \( A^T A \) is always

(Circle the TWO most correct answers.)

i. symmetric.
ii. non-singular.
iii. positive definite.
iv. positive semi-definite.

(c) Suppose that an \( n \times n \) matrix \( A \) is perfectly well-conditioned, i.e., the condition number \( \kappa(A) = 1 \) in the \( L_\infty \)-norm. Which of the following matrices would then necessarily share this same property?

(Circle the TWO most correct answers.)

i. \( cA \), where \( c \) is any non-zero scalar.
ii. \( BA \), where \( B \) is any non-singular matrix.
iii. \( A^{-1} \), the inverse of \( A \).
iv. \( DA \), where \( D \) is a non-singular diagonal matrix.

(d) What is the condition number of the following matrix using the \( L_1 \)-norm?

(Circle the ONE most correct answer.)

\[
\begin{bmatrix}
4 & 0 & 0 \\
0 & -6 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

i. 2
ii. 3
iii. 1
iv. 12
2. [18% = 3 questions × 6% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

(a) Suppose you have already solved the $n \times n$ linear system $Ax = b$ by $LU$ factorization. What is the further cost (order of magnitude will suffice) of solving a new system

- With the same matrix $A$, but a different right hand side vector $b'$?
- With a different matrix $A'$, but the same right hand side vector $b$?

(b) Suppose the second column of the matrix $M_1A$ during the $LU$ decomposition algorithm is as follows:

\[
a_2 = \begin{bmatrix}
3 \\
2 \\
-1 \\
4
\end{bmatrix}
\]

Specify the elimination matrix $M_2$ that zeros out the last two components of $a_2$.

(c) If $A$ and $B$ are $n \times n$ matrices, with $A$ non-singular and $c \in \mathbb{R}^n$, how would you efficiently compute the product $A^{-1}Bc$? ($A$ need not be diagonally dominant.)
3. [16%] Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
1 & 3 & 2 \\
\end{bmatrix}
\]

(a) Show that \( A \) is singular.

(b) If \( b = (2, 4, 6)^T \), how many solutions are there to the system \( Ax = b \)?

**Hint:** For part (b), you just need to show if \( b \in \text{range}(A) \). You are not required to compute the entire solution set. So the answer should be either zero or infinite.
4. [12%] Let $A$ be a rectangular $m \times n$ matrix with linearly independent columns, where $m > n$, and $b \in \mathbb{R}^m$. Show that for every $x \in \mathbb{R}^n$, we have

$$\|Ax - b\|_2^2 = \|A(x - x_0)\|_2^2 + \|Ax_0 - b\|_2^2$$

where $x_0 \in \mathbb{R}^n$ is the least squares solution of $Ax = b$.

**Hint:** Use the identity $\|x\|_2^2 = x^Tx$. 
5. [10%] Prove that the matrix

\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

has no \( LU \) factorization, i.e., no lower triangular matrix \( L \) and upper triangular matrix \( U \) exist, such that \( A = LU \).

**Hint:** Assume that there is an \( LU \) decomposition

\[
\begin{bmatrix}
  l_{11} & 0 \\
  l_{21} & l_{22}
\end{bmatrix}
\begin{bmatrix}
  u_{11} & u_{12} \\
  0 & u_{22}
\end{bmatrix}
= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and arrive at a contradiction!
6. [20%] Consider the $n \times n$ linear system $Ax = b$.

(a) Show that if $A$ is diagonal, the Jacobi method converges after just one iteration.
(b) Show that if $A$ is lower triangular, the Gauss-Seidel method converges after just one iteration.

**Hint:** Decompose $A = D - L - U$ and use the definitions of the Jacobi and Gauss-Seidel methods, as described in the notes.