CS428 Graphics Lecture Monday 03/04/2019

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1 Rotations

- NOTE: 3x3 matrices have 9 numbers but only 3 are independent

Euler Angles Have Singularities (Inverse problem cannot always be solved, however)

ZYZ	Refers to rotation angle about current frame
RPY	Refers to rotation angle about fixed frame
Angle And Axis	A general 3D $\vec{v}$ and angle α

2 Unit Quaternions

- Using Quaternions, the inverse problem can always be solved.
- Unit Quaternions represented by 4 numbers

$Q = (\ \zeta, \vec{\varepsilon}\ )$, $\zeta = \cos(\frac{v}{2})$, $\varepsilon = sin(\frac{v}{2})\vec{\gamma}$

v = angle of rotation, γ = axis of rotation

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

Assume $\vec{\gamma}$ is a unit vector then $\vec{\varepsilon} = sin(\frac{v}{2})\vec{\gamma}$

$$||\vec{\gamma}||^2 = 1$$

$$\gamma_x^2 + \gamma_y^2 + \gamma_x^2 = 1$$

$$\vec{\varepsilon} = sin(\frac{v}{2})\vec{\gamma},\ ||\vec{\varepsilon}||^2 = \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 = sin^2(\frac{v}{2})(\gamma_x^2 + \gamma_y^2 + \gamma_z^2) = sin^2(\frac{v}{2})$$

$$\zeta^2 + ||\vec{\varepsilon}||^2 = cos^2(\frac{v}{2}) + sin^2(\frac{v}{2}) = 1$$

$$Q = (\ \zeta, \vec{\varepsilon}\ ) \quad \boxed{\zeta^2 + \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 = 1} \leftarrow \text{Unit Quaternion}$$

2.1 Rotation Matrix With Quaternion

$$R(\ \zeta, \vec{\varepsilon}\ ) = \begin{bmatrix} 2(\zeta^2 + \varepsilon_x^2) - 1 & 2(\varepsilon_x \varepsilon_y - \zeta \varepsilon_z & 2(\varepsilon_x \varepsilon_z + \zeta \varepsilon_y) \\ 2(\varepsilon_x \varepsilon_y + \zeta \varepsilon_z) & 2(\zeta^2 + \varepsilon_y^2) - 1 & 2(\varepsilon_y \varepsilon_z - \zeta \varepsilon_x) \\ 2(\varepsilon_x \varepsilon_z - \zeta \varepsilon_y) & 2(\varepsilon_y \varepsilon_z + \zeta \varepsilon_x) & 2(\zeta^2 + \varepsilon_z^2) - 1 \end{bmatrix}$$

(if you represent the rotation as a quaternion)

3 Inverse Problem: Construct rotation matrix and convert it to a quaternion

$$R = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$\zeta = \frac{1}{2}\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33} + 1}$

$$\vec{\varepsilon} = \frac{1}{2} \begin{bmatrix} sin(\gamma_{32} - \gamma_{23})\sqrt{\gamma_{11} - \gamma_{22} - \gamma_{33} + 1} \\ sin(\gamma_{13} - \gamma_{31})\sqrt{\gamma_{22} - \gamma_{33} - \gamma_{11} + 1} \\ sin(\gamma_{21} - \gamma_{12})\sqrt{\gamma_{33} - \gamma_{11} - \gamma_{22} + 1} \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

- Note: $sin(x) = 1$, for $x >= 0$, and $sin(x) = -1$, for $x < 0$

We always use quaternions in graphics pipeline

$Q \equiv (\ \zeta, \vec{\varepsilon}\ ) \equiv R(\ \zeta, \vec{\varepsilon}\ )$
$RR^T = I \rightarrow R^{-1} = R^T$
$R^{-1}(\ \zeta, \vec{\varepsilon}\ ) \equiv Q^{-1}(\ \zeta, \vec{\varepsilon}\ ) \equiv (\ \zeta, -\vec{\varepsilon}\ )$

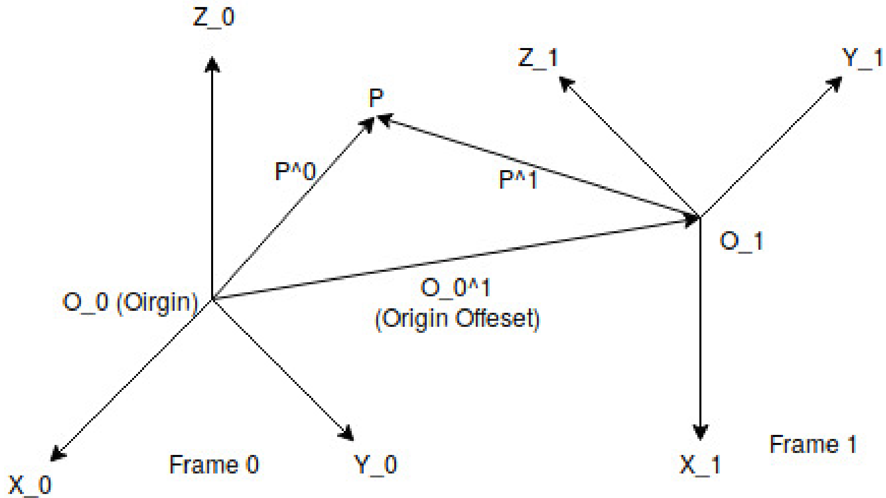
$Q_1(\ \zeta_1, \vec{\varepsilon}_1) \equiv R_1(\ \zeta_1, \vec{\varepsilon}_1\ )$
$Q_2\ \zeta_2, \vec{\varepsilon}_2) \equiv R_2(\ \zeta_2, \vec{\varepsilon}_2\ )$

$R_1 * R_2 \equiv R_1 R_2$

$$\boxed{Q_1 * Q_2 \equiv (\ \zeta_1 \zeta_2 - \varepsilon_1^T \varepsilon_2),\ \ \zeta_1 \vec{\varepsilon}_2 + \zeta_2 \vec{\varepsilon}_1 + \vec{\varepsilon}_1 * \vec{\varepsilon}_2)}$$

Suppose $Q_2 \equiv Q_1^{-1} \implies Q + Q_2 = Q * Q_1^{-1} = I = (1, \vec{0})$
$Q \equiv (\ \zeta, \vec{\varepsilon}\ ),\ Q^{-1} \equiv (\ \zeta, -\vec{\varepsilon}\ )$
So $\zeta_1 \zeta_2 - \varepsilon_1^T \varepsilon_2 = \zeta^2 + \varepsilon^T \varepsilon = \zeta^2 + \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 = 1$ (by definition of Quaternion)
$\zeta_1 \vec{\varepsilon}_2 + \zeta_2 \vec{\varepsilon}_1 + \vec{\varepsilon}_1 * \vec{\varepsilon}_2 = -\zeta \vec{\varepsilon} + \zeta \vec{\varepsilon} - \vec{\varepsilon} * \vec{\varepsilon} = 0$

4 Homogeneous Transformations



$p^0 = O_1^0 + R_1^0 P^1$
Instead of using matrix vector multiplication and vector-vector addtition we just use matrix-vector multiplication

See shader.vs in the Transformations lecture

$$\hat{P} = \begin{bmatrix} P^1_{[3x1]} \\ 1 \end{bmatrix}_{[4x1]}, \ A_1^0 = \begin{bmatrix} R^0_{1[3x3]} & O^0_{1[3x1]} \\ 0^T & 1_{[1x1]} \\ \varepsilon_z & \end{bmatrix}_{[4x4]}, \ 0^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{[1x3]}$$

$P^0 = O_1^0 + R_1^0 P^1$
$R_1^{0^{-1}} = R_1^{0T} \implies R_1^{0^{-1}} P^0 = R_1^{0T} P^0 = R_1^{0T} O_1^0 + R_1^{0T} R_1^0 P_1, \ R_1^{0T} R_1^0 P_1 = P^1$
$R_1^{0T} P^0 = R_1^{0T} O_1^0 + P^1 \implies P^1 = -R_1^{0T} O_1^0 + R_1^{0T} P^0$

$$A_0^1 = \begin{bmatrix} R_1^{0T} & -R_1^{0T} O_1^0 \\ 0^T & 1 \end{bmatrix} * \begin{bmatrix} P^0 \\ 1 \end{bmatrix}_{(\hat{P}^0)} = \begin{bmatrix} P^1 \\ 1 \end{bmatrix}_{(\hat{P}^1)}$$

$\hat{P}^1 = A_0^1 \hat{P}^0, \ \hat{P}^0 = A_1^0 \hat{P}^1 \ \hat{P}^1 = A_0^1 A_1^0 \hat{P}^0 \implies A_0^1 A_1^0 = I \implies A_0^1 = A_1^{0^{-1}}$

$$A_1^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0^T & 1 \end{bmatrix}, \ A_0 = \begin{bmatrix} R_1^{0T} & -R_1^{0T} O_1^0 \\ 0^T & 1 \end{bmatrix}$$

NOTE $A_0^1 \neq A_1^{0T}$ but $R_0^1 = R_1^{0T}$

5 Manipulator Arm