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Last class we discussed rotation matrices with the general rotation matrix:

$$R=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Which has 9 inputs and 6 previously discussed contraints. Implying that we have only 3 independent variables.
The number of independent variables is known as Degree Of Freedom (DOF).
There are multiple methods to represent rotations by using less angles and they are as follows.

Euler Angles:
The euler angles utilize three rotations in the local coordinate axes of the body, or XYZ. These three variables give us $3^3=27$ permutations of rotations, of which we can only use 12 since we have the constraint that *we don't apply multiple consecutive rotations about the same axis*. This can be proved by listing out the 27 permutations and excluding the ones that don't follow the constraint.

~~XXX~~	~~YXX~~	~~ZXX~~
~~XXY~~	YXY	ZXY
~~XXZ~~	YXZ	ZXZ
XYX	~~YYX~~	ZYX
~~XYY~~	~~YYY~~	~~ZYY~~
XYZ	~~YYZ~~	ZYZ
XZX	YZX	~~ZZX~~
XZY	YZY	~~ZZY~~
~~XZZ~~	~~YZZ~~	~~ZZZ~~

ZYZ Angles:
We consider rotating a body about the following axes and angles. $R_z(\varphi), R_{Y'}(\nu), R_{Z''}(\psi)$
Therefore the rotation matrix takes the form:

$$R(\phi)=R_z(\varphi)R_{Y'}(\nu)R_{Z''}(\psi)=\begin{bmatrix} c_\varphi c_\nu c_\psi - s_\varphi s_\psi & -c_\varphi c_\nu s_\psi - s_\varphi c_\psi & c_\varphi s_\nu \\ s_\varphi c_\nu c_\psi + c_\varphi s_\psi & -s_\varphi c_\nu s_\psi + c_\varphi c_\psi & s_\varphi s_\nu \\ -s_\nu c_\psi & s_\nu s_\psi & c_\nu \end{bmatrix}$$

Where

From here we can consider the inverse problem. Where, given the rotation matrix:

$$R=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

we try to find the angles φ, ν, ψ. And the formulas for doing so are:

$\varphi = atan2(r_{23,} r_{13})$ where atan2 is a function in c++ mathematically equivalent to: $atan2(b,a)=\arctan(\frac{b}{a})$

Taking tan of both sides we get: $\tan(\varphi)=\frac{r_{23}}{r_{13}}$

Similarly:

$$\nu = atan2\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \rightarrow \tan(\nu) = \frac{\sqrt{r_{13}^2 + r_{23}^2}}{r_{33}}$$

$$\psi = atan2\left(r_{32,} - r_{31}\right) \rightarrow \tan(\psi) = \frac{r_{32}}{-r_{31}}$$

We can see that we encounter singularities on some specific values such as $r_{31} = 0$ or $r_{33} = 0$ or $r_{13} = 0$. For that reason the mapping from the angles to the rotation matrix is not one-to-one.

RPY(Roll, Pitch, Yaw) angles:

$$R(\phi) = R_z(\varphi) R_Y(\nu) R_X(\psi) = \begin{bmatrix} c_\varphi c_\nu & c_\varphi s_\nu s_\psi - s_\varphi c_\psi & c_\varphi s_\nu c_\psi + s_\varphi s_\psi \\ s_\varphi c_\nu & s_\varphi s_\nu s_\psi + c_\varphi c_\psi & s_\varphi s_\nu c_\psi - c_\varphi s_\psi \\ -s_\nu & c_\nu s_\psi & c_\nu c_\psi \end{bmatrix}$$

Again we can consider the inverse problem, where given the rotation matrix we try to find the angles:

$$\varphi = atan2\left(r_{21,} r_{11}\right) \rightarrow \tan(\varphi) = \frac{r_{21}}{r_{11}}, \text{ singularity at } r_{11} = 0$$

$$\nu = atan2\left(-r_{31,} \sqrt{r_{32}^2 + r_{33}^2}\right) \rightarrow \tan(\nu) = \frac{-r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}}, \text{ singularity at } r_{32}^2 + r_{33}^2 = 0$$

$$\psi = atan2\left(r_{32,} r_{33}\right) \rightarrow \tan(\psi) = \frac{r_{32}}{r_{33}}, \text{ singularity at } r_{33} = 0$$

Angle and Axis rotation:

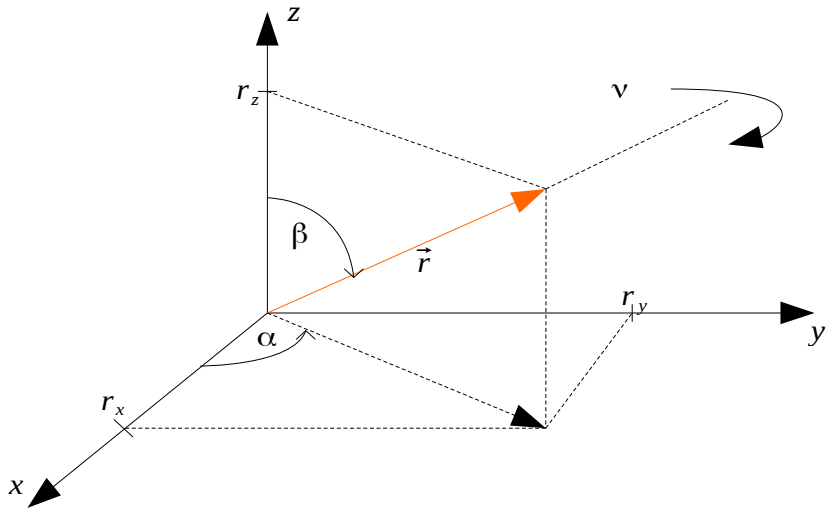
In this case the rotation is defined by a custom axis that we rotate by and the angle ammount that we rotate. $R(\nu, \vec{r})$ where ν is the angle and $\vec{r}$ is the axis of rotation as a unit vector

To perform a rotation using this scheme we follow these steps:

1. Align $\vec{r}$ with the z axis (x or y are also a valid choice). To achieve this alignment you rotate $-\alpha$ about z and $-\beta$ along y

2. Rotate by ν degrees along z

3. Undo step 1. Rotate by β about y and α along z



Identities that follow this transformation:

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_z^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z$$

Inverse Problem:

Given $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ find the angle and rotation:

$$\nu = \arccos\left(\frac{1}{2} * (r_{11} + r_{22} + r_{33} - 1)\right)$$

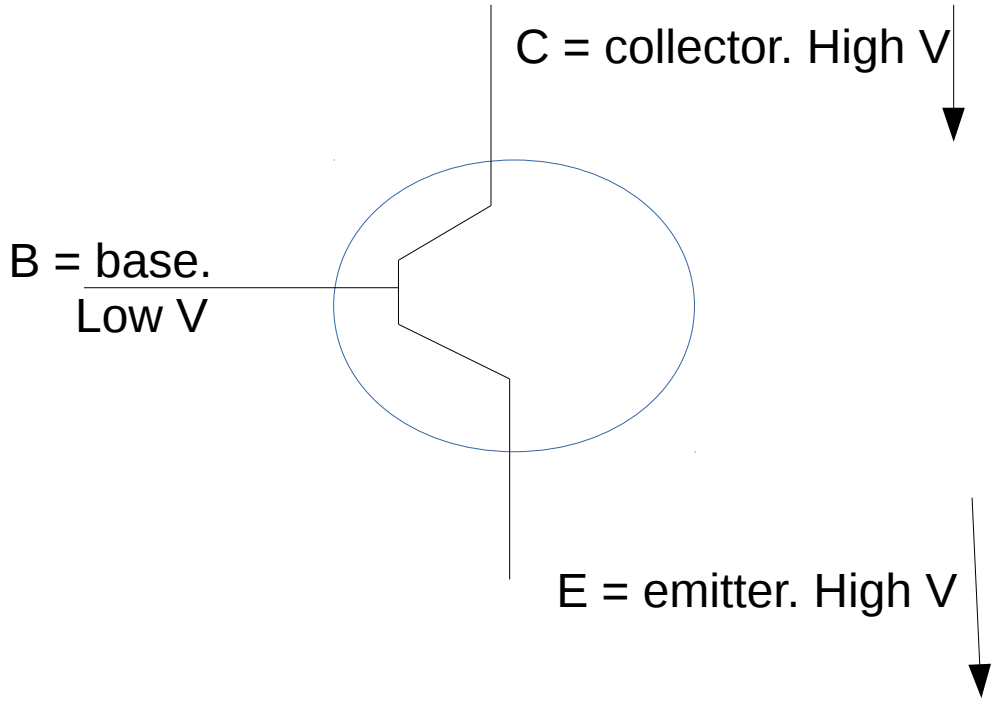
$$\vec{r} = \frac{1}{2\sin\nu} * \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$r_x^2 + r_y^2 + r_z^2 = 1$ → *unit vector*

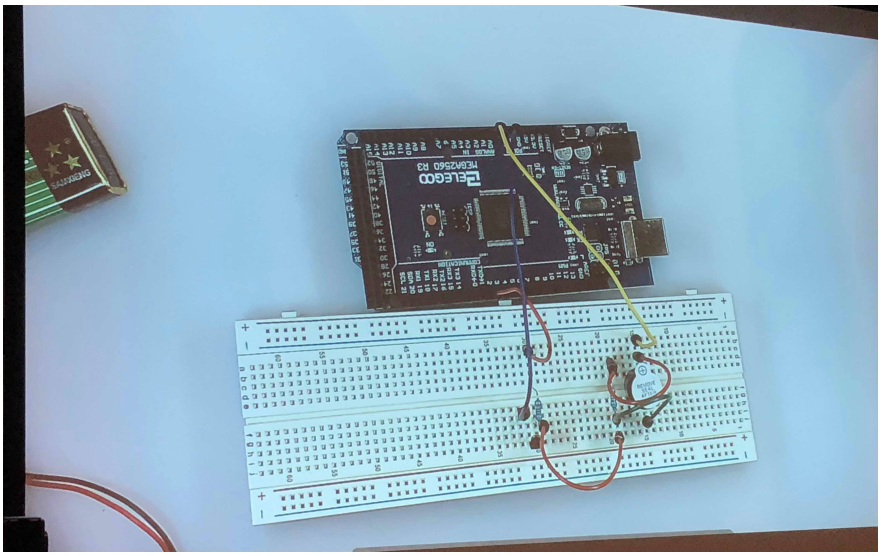
Hardware portion of the class:

The professor explained that the included 9V battery can be used to power the arduino after you've uploaded your logic to it and that way you can use it untethered.

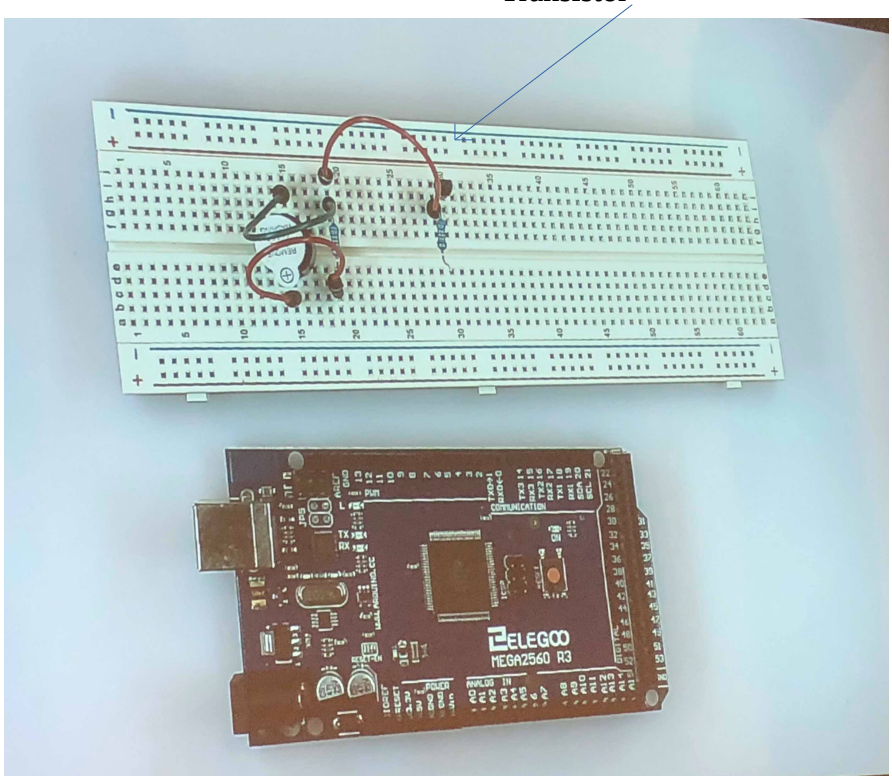
He also explained and demonstrated the use of a transistor.



The professor’s demonstration had this circuit:



Transistor



Along with the following code:

#define PIEZ0 3
int del = 500;

```
void setup(){
    pinMode(PIEZ0, OUTPUT);
}

void loop() {
    analogWrite(PIEZ0, 128);
    delay(del);
    digitalWrite(PIEZ0, LOW);
    delay(del);
}
```

This code along with the circuit periodically produced sound.
In order to tell which leg is which on our included transistors, we look at the leg with a number on it and that leg is the collector (C). The middle leg is the base (B) and the remaining leg is the emitter (E).