2/25 Lecture

Rotation continued

Degrees of freedom (DoF): rotation matricies only have 3 DoFs which are essentially the number of components not restricted by the various assumptions we make to construct the rotation matrix.

Euler angles: angles that describe our rotation in the real space (alternatively we have Quaternion angles which lie in complex space).
With 3 axes, there are 27 possible rotation permutations possible, assuming we are making 3 rotations. But, it turns out we only need 12 of them because we can't rotate on the same axes consecutively.

Our 12 possible rotations:

xyx	yxy	zxy
xyz	yxz	zxz
xzx	yzx	zyx
xzy	yzy	zyz

Before discussing the various rotation systems that exist for Euler angles, first a reminder that the general rotation matrix is of the form
R =

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

ZYZ Angles

$R(\phi) = R_z(\varphi), R'_y(\nu), R''_z(\psi)$ is the equation of our rotation about ϕ along the ZYZ pair of axes.

(NOTE: For brevity, we will abbreviate cosine as just "c" and sine as "s")

Our rotation $R(\phi)$ is the following,

$R(\phi) =$

$$\begin{bmatrix} c_\varphi c_\nu c_\psi - s_\varphi s_\psi & -c_\varphi c_\nu c_\psi - s_\varphi s_\psi & c_\varphi s_\nu \\ s_\varphi c_\nu c_\psi + c_\varphi s_\psi & -s_\varphi c_\nu c_\psi + c_\varphi s_\psi & s_\varphi s_\nu \\ -s_\nu c_\psi & s_\nu s_\psi & c_\nu \end{bmatrix}$$

Inverse (ZYZ)

Our inverse functions will produce the angles φ, ν, and ψ. We obtain them by applying inverse trig functions on the various elements of our general rotation matrix, R.

$\varphi = atan2(r_{23}, r_{13}) = arctan(r_{23}/r_{13})$ (NOTE: atan2 is the function provided by some math libraries. It is the function for arctan)

$\nu = atan2(\sqrt{r_{13}^2 + r_{23}^2}, r_{33})$

$\psi = atan2(r_{32}, -r_{31})$.

Our inverse mappings are NOT one-to-one (1-1). Meaning, we will have some angles that are singularities which our representation of rotation along ZYZ cannot represent.

RPY Angles (Roll, Pitch, Yaw angles – rotation with respect to a fixed axis)

Let $\phi = (\varphi, \nu, \psi)$ and $R_x(\psi), R_y(\nu), R_z(\varphi)$ be the corresponding rotation matrices.
Then, we have

$R(\phi) = R_x(\psi), R_y(\nu), R_z(\varphi) =$

$$\begin{bmatrix} c_\varphi c_\nu & c_\varphi c_\nu c_\psi - s_\varphi s_\psi & c_\varphi c_\nu c_\psi + s_\varphi s_\psi \\ s_\varphi s_\nu & s_\varphi s_\nu s_\psi + c_\varphi c_\psi & s_\varphi s_\nu s_\psi - c_\varphi c_\psi \\ -s_\nu & c_\nu s_\psi & c_\nu c_\psi \end{bmatrix}$$

Inverse (RPY)
Again, we have similar issues of singularity degeneracy in the RPY system. Our inverse functions are as follows,
$\varphi = atan2(r_{21}, r_{11})$

$\nu = atan2(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$

$\psi = atan2(r_{32}, r_{33})$

Angle and Axis (rotation about a particular axis) The idea of this rotation is rotation about a particular vector and not just strictly an axis.
Our rotation is given by R$(\nu, \vec{r})$ where $\vec{y} = (r_x, r_y, r_z)$. But before we can specify our rotation, we first have to ensure of a few things:

1. Align our vector $\vec{r}$ with respect to z by rotation -α about z, -β about y, assuming our vector was initially α and β away from z.

2. Rotate by ν along z

3. Rotate by β about y, α about z (the order of rotation matters)

Our rotation is then R(ν, $\vec{r}$) = $R_z(\alpha)R_y(\beta)R_z(\nu)R_y(-\beta)R_z(-\alpha)$ (actual rotation matrix omitted during lecture).
And with this we have,

$\sin\alpha = r_y/\sqrt{r_x^2 + r_y^2}$
$\cos\alpha = r_x/\sqrt{r_z^2 + r_y^2}$
$\sin\beta = \sqrt{r_x^2 + r_y^2}$
$\cos\beta = r_z$

These formulas are made under the assumption that our vector $\vec{r}$ is a unit vector.

Inverse (Angle and Axis)
$\nu = arccos((r_{11} + r_{22} + r_{33} - 1)/2)$
$\vec{r} = 1/(2sin\nu) \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$
These formulas are under the assumption that r is a unit vector.
Again, angle and axis rotation method also has degeneracy issues with singularities.

Unit Quaternions Quaternions are angles based on the complex numbers rather than real numbers but most importantly quaternion angles don't suffer from issues with singularities and there exist 1-1 mappings between quaternions and euler angles.

Arduino – Quick note on transistors

Our Arduino boards come with a 9V battery, which is more than what is recommended for any piece that comes with our Arduinos. We feed an appropriate amount of current to our pieces through transistors.
Meanwhile, we can use the batteries to power our Arduinos instead of relying on our computers.

How do transistors work? They're a 1-way circuit that is essentially a switch. We have a collector end (which collects current) and an emitter end (which emits current). We connect the collector and emitter so current can flow through the base which needs a current to be sent through it (it's basically a switch that works with current rather than physical press).